# $[Q, R] = 0$ and Verlinde Series

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Joint with Eckhard Meinrenken

# The Hamiltonian $[Q, R] = 0$ Theorem

- $G$ compact Lie group, Lie algebra $\mathfrak{g}$, representation ring $R(G)$
- $\phi : M \to \mathfrak{g}^*$ compact Hamiltonian $G$-space
- $L \to M$ prequantum line bundle, $G$-equivariant
- $D_L$ Dolbeault-Dirac operator for compatible almost $\mathbb{C}$ structure, twisted by $L$
- $Q(M) \in R(G)$ the index of $D_L$

**Theorem (Meinrenken) "Guillemin-Sternberg principle"**

Assume 0 is a regular value. Then

$$Q(M)^G = Q(M//G) \qquad \left(\text{where } M//G = \phi^{-1}(0)/G\right)$$

# Approaches of Paradan and Szenes-Vergne

- Paradan (2001) decomposed the index $Q(M) \in R(G)$ into a sum of simpler, infinite-dimensional contributions $Q_\beta(M) \in R^{-\infty}(G)$ indexed by the components of the critical set of $||\phi||^2$.
- Szenes-Vergne (2010) re-derived Paradan's decomposition using a combinatorial rearrangement of the Atiyah-Bott fixed-point formula, together with asymptotics of $Q(M, L^k)$, $k \to \infty$.

# Example

$$M = \mathbf{CP}^1 \circlearrowleft S^1 = G,\ t \cdot [x : y] = [tx : t^{-1}y],\ k > 0$$

$$Q(M, L^k)(t) = \frac{t^k}{1 - t^{-2}} + \frac{t^{-k}}{1 - t^2} = \sum_{n=0}^{k} t^{-k+2n}.$$

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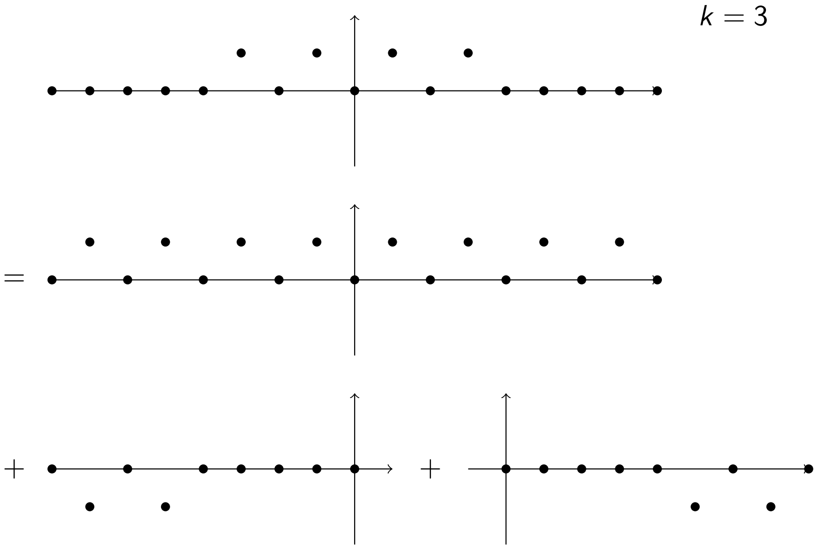
Paradan's formula in terms of multiplicities:

$$\begin{aligned} m(\lambda, k) &= \begin{cases} 1 & |\lambda| \leq k, \lambda = k \ (mod\, 2) \\ 0 & \text{else} \end{cases} \\ &= m_{\text{q-pol}}(\lambda, k) + m_+(\lambda, k) + m_-(\lambda, k) \end{aligned}$$

$m_\pm$ are corrections, supported away from $0 \in \mathbb{Z} = \Lambda^*$.

# Example



# Quasi-polynomials

Let $\Gamma$ be a lattice.

**Definition**

A function $f : \Gamma \to \mathbb{C}$ is **quasi-polynomial** if there is a full-rank sublattice $\Gamma' \subset \Gamma$ such that $f$ is polynomial on each coset $\gamma + \Gamma'$.

**Example**

$$m_{\text{q-pol}}(\lambda, k) = \begin{cases} 1 & \lambda = k \ (mod\, 2) \\ 0 & \text{else} \end{cases}$$

is quasi-polynomial on $\Lambda^* \times \mathbb{Z}$.

Important! Study multiplicities as function of $(\lambda, k) \in \Lambda^* \times \mathbb{N}$.

# Restriction to $T$

**Weyl character formula**

$$\chi_\lambda|_T = \frac{\sum_w (-1)^{\ell(w)} e_{w(\lambda+\rho)-\rho}}{\prod_{\alpha>0} 1 - e_{-\alpha}}, \qquad \lambda \in \Lambda^* \cap \mathfrak{t}^*_+$$

Multiplicity function $m$ for $Q(M) \in R(G)$

$$m : \Lambda^* \cap \mathfrak{t}^*_+ \to \mathbb{Z}$$

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Instead study

$$\prod_{\alpha>0} (1 - e_{-\alpha}) Q(M)|_T \in R(T)$$

$\rightarrow$ multiplicity function $m^a$ is W-anti-symmetric (shifted action), and

$$m^a(\lambda) = m(\lambda), \qquad \lambda \in \Lambda^* \cap \mathfrak{t}^*_+$$

# Interpretation

$\phi : M \to \mathfrak{g}^*$ transverse to $\mathfrak{t}^*$ $\Rightarrow$

$$\prod_{\alpha>0}(1 - e_{-\alpha})Q(M)|_T = Q(X), \qquad X = \phi^{-1}(\mathfrak{t}^*)$$

# Hamiltonian $LG$-spaces

- $LG = \mathrm{Map}(S^1, G)$ loop group, $L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$
- $L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$, $LG$ acts by gauge transformations:

$$g \cdot \xi = \mathrm{Ad}_g \xi - dgg^{-1}.$$

**Definition**

A *Hamiltonian $LG$-space* $(\mathcal{M}, \omega, \Psi)$ consists of a symplectic (Banach) $LG$-manifold, equipped with a proper moment map $\Psi : \mathcal{M} \to L\mathfrak{g}^*$.

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**Example: Moduli of flat connections**

$\Sigma$ compact Riemann surface with $b$ boundary components $\Rightarrow$

$$\mathcal{M} = \mathcal{A}^{\mathrm{flat}}(\Sigma)/\{g \in \mathcal{G} : g|_{\partial\Sigma} = 1\}, \quad \Psi(A) = \iota^*_{\partial\Sigma} A$$

is a Hamiltonian $LG^b$-space.

# Quantization

Let $L \to \mathcal{M}$ be a prequantum line bundle ($\widehat{LG}$-equivariant).

$\exists$ "quantization" method (Meinrenken '10) for the finite-dimensional quotient:

$$M = \mathcal{M}/\Omega G$$

($M$ no longer symplectic, instead **quasi-Hamiltonian** (AMM '98))

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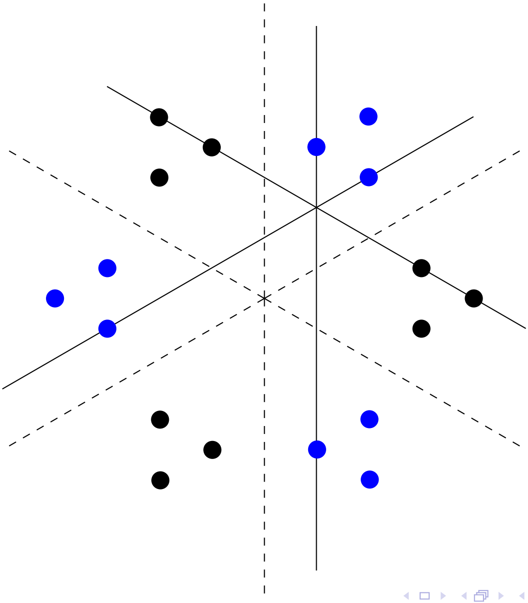
The analog of $m^a$ exists... It is a function:

$$m^a : \Lambda^* \to \mathbb{Z}$$

which is $\mathbf{W}_{\text{aff}}$**-anti-symmetric** (shifted). Take powers $L^k \Rightarrow$

$$m^a : \Lambda^* \times \mathbb{N} \to \mathbb{Z}$$

# $m^a$ for a Hamiltonian $LSU(3)$-space



# Multiplicities

**Formula for $m^a$**

$$m^a(\lambda, k) = \frac{1}{|T_\ell|} \sum_{t \in T_\ell} t^{-\lambda} \prod_{\alpha > 0} (1 - t^{-\alpha}) Q(M, k)(t)$$

$$Q(M, k)(t) = \sum_{F \subset M^t} \int_F \frac{\hat{\mathrm{A}}(F) \mathrm{Ch}(\mathcal{L}_k, t)^{1/2}}{\mathcal{D}_{\mathbb{R}}(\nu_F, t)}$$

- $\ell = k + h^\vee$, $h^\vee$ dual Coxeter number (e.g. $h^\vee = N$ for $SU(N)$)
- $T_\ell$ finite "Verlinde" subgroup

$$T_\ell := \ell^{-1} B^\sharp(\Lambda^*)/\Lambda \subset \mathfrak{t}/\Lambda = T$$

# Plan

Goal:

Prove $m^a(0,k)$ is **quasi-polynomial**, using strategy inspired by Szenes-Vergne-Paradan. Derive $[Q,R]=0$ from this + stationary phase argument for large $k$.

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- Finite group $T_\ell \subset T$, varying with $\ell$.

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Challenges:

- Finite group $T_\ell \subset T$, varying with $\ell$.
- $||\Psi||^2$ on $\mathcal{M}$ does not descend to $M$.

# Verlinde series

$\Lambda \subset \Xi$ lattices in vector space $\mathfrak{h}$
$\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$, $\alpha_k \in \Lambda^*$.

**Definition**

The *Verlinde series* $V(\boldsymbol{\alpha}) : \Lambda^* \times \mathbb{N} \to \mathbb{C}$ is

$$V(\boldsymbol{\alpha})(\lambda, \ell) = \sum_{\xi \in \ell^{-1}\Xi/\Lambda}{}' \frac{e^{-2\pi i \langle \lambda, \xi \rangle}}{\prod_{\boldsymbol{\alpha}} 1 - e^{2\pi i \langle \alpha, \xi \rangle}}.$$

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Szenes' ('03): remarkable residue formula for $V(\boldsymbol{\alpha})$. Implies $V(\boldsymbol{\alpha})$ is **piecewise quasi-polynomial**.

$\Rightarrow$ for generic $\gamma$, $V(\boldsymbol{\alpha})$ has a **quasi-polynomial "germ"** $\mathrm{Ver}(\boldsymbol{\alpha}, \gamma)$.

# Expanding in different directions

Expansion in direction $+1$:

$$\frac{1}{1-t} = \sum_{n\geq 0} t^n$$

Expansion in direction $-1$:

$$\frac{1}{1-t} = \frac{-t^{-1}}{1-t^{-1}} = -t^{-1} \sum_{n\geq 0} t^{-n}$$

# A combinatorial decomposition

$\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$. Choose generic vector $\gamma^+$, and define $P(\boldsymbol{\alpha}, \gamma^+)(\lambda)$ to be the coefficient of $t^\lambda$ in expansion of

$$\frac{1}{\prod_k (1 - t^{\alpha_k})}$$

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**Theorem (Boysal-Vergne)**

*Let $\mathcal{S}$ be the set of affine subspaces generated by $\Xi^*$, $\boldsymbol{\alpha}$. For generic $\gamma$,*

$$V(\boldsymbol{\alpha}) = \sum_{\Delta \in \mathcal{S}} \mathrm{Ver}(\boldsymbol{\alpha}_\Delta, \gamma_\Delta) * P(\boldsymbol{\alpha}'_\Delta, \gamma_\Delta^+)$$

*where $\gamma_\Delta = \mathrm{proj}_\Delta(\gamma)$, $\gamma_\Delta^+ = \gamma_\Delta - \gamma$ and $\boldsymbol{\alpha}_\Delta \cup \boldsymbol{\alpha}'_\Delta = \boldsymbol{\alpha}$.*

# Rearrangement of fixed-point formula I

Recall $T_\ell = \ell^{-1} B^\sharp(\Lambda^*)/\Lambda$, and

$$m^a(\lambda, k) = \frac{1}{|T_\ell|} \sum_{t \in T_\ell} t^{-\lambda} \prod_{\alpha > 0} (1 - t^{-\alpha}) \sum_{F \subset M^t} \int_F \frac{\hat{A}(F)\mathrm{Ch}(\mathcal{L}_k, t)^{1/2}}{\mathcal{D}_{\mathbb{R}}(\nu_F, t)}.$$

**Lemma**

*The multiplicity $m^a$ is a sum of a (possibly large number) of* **translates** *of Verlinde series, for various infinitesimal stabilizers $\mathfrak{h} \subset \mathfrak{t}$, weighted by characteristic numbers of the fixed-point sets.*

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Apply Boysal-Vergne formula term-by-term $\rightarrow$ complicated formula for $m^a$.

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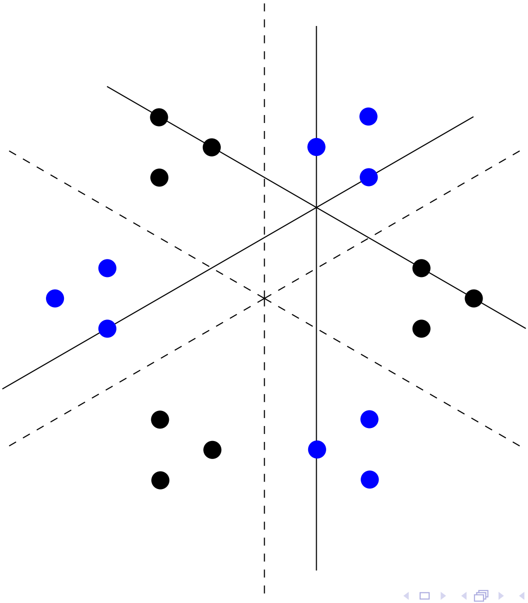

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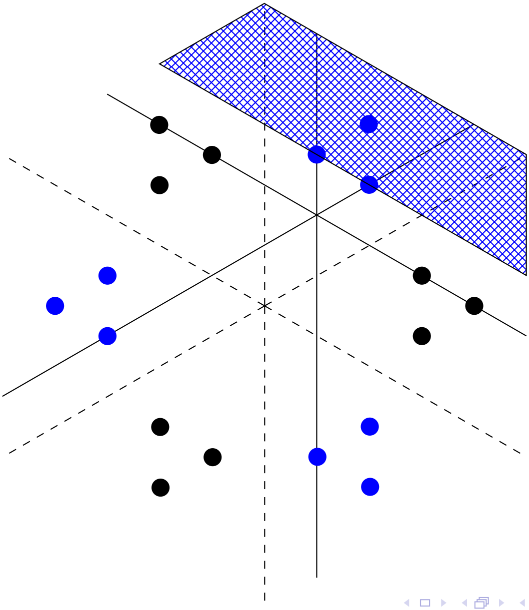

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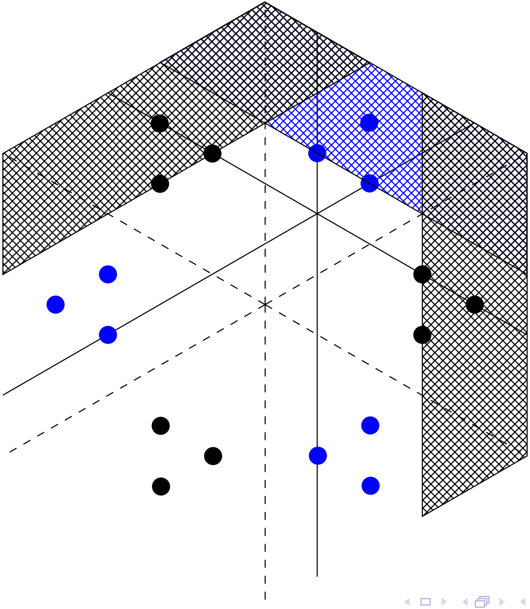

# Rearrangement of fixed-point formula II

Schematically:

$$m^a = m^a_{\text{q-pol}} + \sum_{\Delta} m^a_{\Delta},$$

where $\Delta \in \mathcal{S}(M)$ a collection of proper affine subspaces of $\mathfrak{t}^*$.

Each term $m^a_\Delta$ involves a further sum over fixed-point sets $F \subset M^{\mathfrak{t}_\Delta}$, $\mathfrak{t}_\Delta = \Delta^\perp$.

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Moreover:

- central contribution $m^a_{\text{q-pol}}$ is quasi-polynomial
- each $m^a_\Delta(-, k)$ supported in a half-space
- for $k >> 0$, these half-spaces do not contain 0

# Quasi-polynomial?

$$m^a = m^a_{\text{q-pol}} + \sum_{\Delta} m^a_{\Delta}$$

**If we show**

$$0 \notin \operatorname{supp}(m^a_{\Delta}) \quad \textbf{for all} \quad k \geq 1$$

**then**

$$m^a(0,k) = m^a_{\text{q-pol}}(0,k)\ \forall k \geq 1 \quad \Rightarrow \quad m^a \text{ quasi-polynomial!}$$

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- Bound translations ("rho shifts") occuring above.
- Prove remaining problematic terms (those not corresponding to critical points of $||\Psi||^2$) **cancel out**.

# An inequality

The Stiefel diagram is an affine hyperplane arrangement in $\mathfrak{t}$:

$$\langle \alpha, x \rangle + n = 0, \qquad \alpha \in \mathcal{R}, n \in \mathbb{Z}.$$

$v$ a vertex and $K = G_{\exp(v)}$. Let $\rho$, $\rho_K$ be half-sums of positive roots for $G$, $K$, and let

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**Theorem**: Identify $\mathfrak{t} \simeq \mathfrak{t}^*$ (basic inner product). Then

$$||v - c_K|| \geq ||c_K||.$$

Special case: $G = SU(N)$, then $c_K = c$ is the barycenter of the alcove.

# Cancellations

$$m^a = m^a_{\text{q-pol}} + \sum_{\Delta} m^a_{\Delta} \tag{1}$$

**Back to $\Psi : \mathcal{M} \to L\mathfrak{g}^*$**

$$\mathrm{Crit}(||\Psi||^2) = G \cdot \bigcup_{\beta \in \mathcal{B}} \mathcal{M}^{\beta} \cap \Psi^{-1}(\beta).$$

$\beta = 0$ corresponds to $m_{\text{q-pol}}$, and $\beta \in \Delta$ corresponds to $m_{\Delta}$.

A priori, there are **additional problematic terms** in (1) that do not correspond to any $\beta \in \mathcal{B}$.

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A priori, there are **additional problematic terms** in (1) that do not correspond to any $\beta \in \mathcal{B}$.

These terms vanish! (Involves using fixed-point formula in reverse on submanifolds $C \subset M^{t_\Delta}$, and asymptotic argument $k \to \infty$.)

# Asymptotic $[Q, R] = 0$ (rough idea)

Atiyah-Segal-Singer fixed-point formula $\to$ Berline-Vergne "delocalized" index formulas, and stationary phase $k \to \infty$ (Meinrenken '94)

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Recall $\Psi : \mathcal{M} \to L\mathfrak{g}^*$. Assume $X = \Psi^{-1}(\mathfrak{t}^*)$ smooth. For $\xi \in \mathfrak{t}$ small, consider

$$\chi(\xi) := \int_X \hat{\mathrm{A}}(X, \xi)\mathrm{Ch}(\mathcal{L}_k, \xi)$$

Integral lattice $\Lambda$ acts on $X$. Integrand periodic except for phase shift:

$$\chi(\xi) = \sum_\lambda e^{2\pi i \ell \langle \lambda, \xi \rangle} \int_{X_0} \hat{\mathrm{A}}(X, \xi)\mathrm{Ch}(\mathcal{L}_k, \xi)$$

where $X_0$ is a fundamental domain.

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**Poisson summation formula** $\Rightarrow \chi$ supported on $\ell^{-1} B^{\sharp}(\Lambda^*)$.

Recall

$$T_\ell = \ell^{-1} B^{\sharp}(\Lambda^*)/\Lambda$$

Localize on $X/\Lambda \Rightarrow$ fixed-point formulas for $Q(M, k)(t)$, $t = \exp(\xi)$.

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Localize on $X/\Lambda \Rightarrow$ fixed-point formulas for $Q(M, k)(t)$, $t = \exp(\xi)$.

Stationary phase argument $\Rightarrow$

$$\int_{\mathfrak{t}} \chi(\xi) d\xi = \int_{\mathfrak{t}} d\xi \int_X \hat{\mathrm{A}}(X, \xi) \mathrm{Ch}(\mathcal{L}_k, \xi)$$

concentrated near $\Psi^{-1}(0)$ as $k \to \infty$.